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ADMISSIBILITY, STABLE UNITS AND CONNECTED COMPONENTS

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Abstract. Consider a reflection from a finitely-complete category $\mathbb{C}$ into its full subcategory $\mathbb{M}$, with unit $\eta : 1_{\mathbb{C}} \to HI$. Suppose there is a left-exact functor U into the category of sets, such that UH reflects isomorphisms and $U(\eta_C)$ is a surjection, for every $C \in \mathbb{C}$. If, in addition, all the maps $\mathbb{M}(T, M) \to \mathbf{Set}(1, U(M))$ induced by the functor UH are surjections, where T and 1 are respectively terminal objects in $\mathbb{C}$ and $\mathbf{Set}$, for every object M in the full subcategory $\mathbb{M}$, then it is true that: the reflection $H \vdash I$ is semi-left-exact (admissible in the sense of categorical Galois theory) if and only if its connected components are "connected"; it has stable units if and only if any finite product of connected components is "connected". Where the meaning of "connected" is the usual in categorical Galois theory, and the definition of connected component with respect to the ground structure will be given. Note that both algebraic and topological instances of Galois structures are unified in this common setting, with respect to categorical Galois theory.

1. Introduction

A reflection $H \vdash I$ from a category $\mathbb{C}$ into its full subcategory $\mathbb{M}$ can be seen as a Galois structure, one in which all morphisms are taken into account. Hence, such a reflection is semi-left-exact (in the sense of [2]) if and only if it is an admissible Galois structure (in the sense of categorical Galois theory). The fundamental theorem of categorical Galois theory states that, for an admissible Galois structure as above, that is, a semi-left-exact reflection into a full subcategory, there is an equivalence $Spl(E,p) \simeq \mathbb{M}^{Gal(E,p)}$, for every effective descent morphism $p : E \to B$ in $\mathbb{C}$, between the full subcategory $Spl(E,p)$ of the comma category $(\mathbb{C} \downarrow B)$, determined by the morphisms split by $p : E \to B$, and the category $\mathbb{M}^{Gal(E,p)}$ of actions of the Galois pregroupoid $Gal(E,p)$ in $\mathbb{M}$ (see [1]). To establish the existence of such equivalences, that is, in order to prove that the reflection is semi-left-exact, it is necessary to show, for every $B \in \mathbb{C}$ and every $(M,g) \in (\mathbb{M} \downarrow I(B))$, that the counit morphism $\varepsilon^B_{(M,g)} : I^B H^B(M,g) \to (M,g)$ is an isomorphism, where $H^B \vdash I^B : (\mathbb{C} \downarrow B) \to (\mathbb{M} \downarrow I(B))$ is the induced adjunction. In the current paper,

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we prove it is enough to show that every $\varepsilon^B_{(T,g)}$ is an isomorphism when T is a terminal object, in order to guarantee semi-left-exactness, provided there is a ("forgetful") functor U from $\mathbb{C}$ into sets, satisfying certain conditions. Such is the case of the two reflections **CompHaus** $\rightarrow$ **Stone**, compact Hausdorff spaces into Stone spaces, and **SGr** $\rightarrow$ **SLat**, semigroups into semilattices, where "connected components are connected" (meaning that the counit morphisms $\varepsilon^B_{(T,g)}$ are all isomorphisms, which amounts to the preservation by the reflector of the "connected component" pullback diagrams). Furthermore, these two examples are known to satisfy a stronger condition than semi-left-exactness. In fact, both reflections **CompHaus** $\rightarrow$ **Stone** and **SGr** $\rightarrow$ **SLat** have stable units (see [1] and [3], respectively). We will also state that such a Galois structure with such a "forgetful" functor does have stable units if and only if "finite products of connected components are connected". A connected component is simply the pullback $C \times_{(\eta_C,\mu)} T$ of a morphism $\mu : T \rightarrow HI(C)$ from a terminal object T along a unit morphism $\eta_C : C \rightarrow HI(C)$. Therefore, in our setting, semi-left-exactness and the stable units property are simplified and the Galois structures can be classified according to the reflection of connected components and its products, respectively.

Besides semi-left-exactness and the stable units property, there is a weaker property and also a stronger one. When the former holds, a reflection is called simple. A reflection where the latter holds is called a localization, meaning that the reflector is left-exact, that is, it preserves finite limits. In our setting, a sufficient condition, for a reflection to be a localization, will be given on the connected components. Also, semi-left-exact and simple reflections are shown to coincide, provided a further condition holds for the left adjoint I.

Finally, the author would like to mention that the results in this paper had their origin in generalizing the proof of Theorem 3 in [3], where it is shown that the reflection of semigroups into semilattices has stable units.[1]

2. Ground Structure

In this section 2, it is given the setting in which all the propositions of the current paper hold.

Consider an adjunction $H \vdash I : \mathbb{C} \rightarrow \mathbb{M}$, with unit $\eta : 1_{\mathbb{C}} \rightarrow HI$, such that the category $\mathbb{C}$ has finite limits and the right adjoint H is a full inclusion of $\mathbb{M}$ in $\mathbb{C}$, i.e., the adjunction is a reflection of the category $\mathbb{C}$ into its full subcategory $\mathbb{M}$. Consider as well a functor $U : \mathbb{C} \rightarrow \mathbf{Set}$ from $\mathbb{C}$ into the category of sets, with the following properties:

(a) U is left exact (i.e., U preserves finite limits);
(b) UH reflects isomorphisms;
(c) every map $U(\eta_C) : U(C) \rightarrow UHI(C)$ is a surjection, for every unit morphism η_C of the reflection above, $C \in \mathbb{C}$;
(d) every map $\mathbb{C}(T, M) \rightarrow \mathbf{Set}(U(T), U(M))$, which is the restriction of the functor U to the hom-set $\mathbb{C}(T, M)$, is a surjection, for any object $M \in \mathbb{M}$, with T a terminal object in $\mathbb{C}$.

Remark 2.1. It is convenient, without no loss of generality, to chosen the unit $\eta : 1_{\mathbb{C}} \rightarrow HI$ so that the counit is an identity $IH = 1_{\mathbb{M}}$.

[1] The property that "connected components are connected", i.e., semi-left-exactness in our setting, was called attainability in [5], in the particular case of semigroups.

Remark 2.2. It is also convenient to assume, without no loss of generality, that T is a terminal object chosen to be in $\mathbb{M}$. In such case, $\mathbb{C}(T, M) = \mathbb{M}(T, M)$ in (d).[2]

Remark 2.3. Suppose UH has a left adjoint F, being the counit morphism of such an adjunction $\delta : F(UH) \to 1_{\mathbb{M}}$. If the counit morphism of a terminal object $\delta_T : F(UH)(T) \to T$ is a split monomorphism then condition (d) necessarily holds. Notice that all functors UH, considered in any instance of the ground structure presented in last section 8, have a left adjoint, and the respective counit morphisms δ_T of terminal objects are isomorphisms, i.e., F preserves the terminal objects in **Set**.[3]

3. Properties of the Reflection

It is to be defined when the reflection $I \dashv H$ is 1. *simple*, 2. *semi-left-exact* or 3. *to have stable units* (notions introduced in [2]). One easily checks from the definitions below that if I is a left-exact functor, in which case the reflection is called a *localization*, then 1., 2. and 3. hold, and that 3. is stronger than 2., which in turn is stronger than 1. (I is left exact $\Rightarrow I \dashv H$ has stable units $\Rightarrow I \dashv H$ is semi-left-exact $\Rightarrow I \dashv H$ is simple). The semi-left-exactness is also called *admissibility* in categorical Galois theory (see [1]).

Definition 3.1. The reflection $I \dashv H$ is called simple if the morphism $I(w) : I(A) \to I(C)$ is an isomorphism in every diagram of the form

(1)
$$\begin{tikzcd}
A \arrow[dr, "w"] \arrow[drr, "\eta_A"] \arrow[ddr, "f"'] & & \\
& C \arrow[r] \arrow[d] & HI(A) \arrow[d, "HI(f)"] \\
& B \arrow[r, "\eta_B"] & HI(B)
\end{tikzcd} ,$$

where the rectangular part of the diagram is a pullback square, η_A and η_B are unit morphisms, and w is the unique morphism which makes the diagram commute.

Remark 3.1. The functor between comma categories $I^B : (\mathbb{C} \downarrow B) \to (\mathbb{M} \downarrow I(B))$, sending $f : A \to B$ to $I(f)$, has a right adjoint H^B sending $g : M \to I(B)$ to its pullback along $\eta_B : B \to HI(B)$, for each $B \in \mathbb{C}$. Hence, $I \dashv H$ is simple if and only if $I^B \eta^B$ is an isomorphism for every $B \in \mathbb{C}$, where η^B is the unit of the adjunction $I^B \dashv H^B$ (equivalently, $\varepsilon^B I^B$ is an isomorphism for every $B \in \mathbb{C}$, where ε^B is the counit of $I^B \dashv H^B$).

Definition 3.2. The reflection $I \dashv H$ is called semi-left-exact, or admissible, if the left adjoint I preserves all pullback squares of the form

[2]Recall that a full reflective subcategory $\mathbb{M}$ of $\mathbb{C}$ is closed for limits in $\mathbb{C}$.

[3]Notice that any counit morphism is an isomorphism if it is a monomorphism, provided the right adjoint reflects isomorphisms.

(2)
$$\begin{tikzcd}
C \times_{HI(C)} M \arrow[r, "\pi_2"] \arrow[d, "\pi_1"'] & M \arrow[d, "g"] \\
C \arrow[r, "\eta_C"] & HI(C)
\end{tikzcd} ,$$

where the bottom arrow η_C is a unit morphism, and the object M, in the upper corner to the right, is in the subcategory $\mathbb{M}$.

Remark 3.2. The reflection $I \dashv H$ is semi-left-exact if and only if the functor I preserves all pullback squares in which the arrow in the right edge is in the subcategory $\mathbb{M}$, as it is easy to prove. Equivalently, $I \dashv H$ is semi-left-exact if and only if the right adjoint H^B is fully faithful (ε^B is an isomorphism) for every $B \in \mathbb{C}$. Therefore, the reflection is simple if it is semi-left-exact (cf. remark 3.1).

Definition 3.3. The reflection $I \dashv H$ has stable units if the left adjoint I preserves all pullback squares of the form

(3)
$$\begin{tikzcd}
C \times_{HI(C)} D \arrow[r, "\pi_2"] \arrow[d, "\pi_1"'] & D \arrow[d, "g"] \\
C \arrow[r, "\eta_C"] & HI(C)
\end{tikzcd} ,$$

in which the bottom arrow η_C is a unit morphism.

Remark 3.3. One could also show that the reflection $I \dashv H$ has stable units if and only if the left adjoint I preserves all pullback squares in which the object at the right corner in the bottom belongs to the subcategory $\mathbb{M}$.

4. Admissibility and Connected Components

Definition 4.1. Consider any morphism $\mu : T \to HI(C)$ from a terminal object T into $HI(C)$, for some $C \in \mathbb{C}$.

The connected component of the morphism μ, with respect to the ground structure of section 2, is the pullback $C_\mu = C \times_{HI(C)} T$ in the following pullback square

(4)
$$\begin{tikzcd}
C_\mu \arrow[r, "\pi_2^\mu"] \arrow[d, "\pi_1^\mu"'] & T \arrow[d, "\mu"] \\
C \arrow[r, "\eta_C"] & HI(C)
\end{tikzcd} .$$

The following Theorem 4.1 states that, under the assumptions given in section 2, in order to prove the semi-left-exactness of the full reflection $I \dashv H$, one has only to establish the preservation by I of the pullback squares like those in diagram (2) in which the object M is terminal. So, in our context, semi-left-exactness reduces to connected components being "connected", in the sense $HI(C_\mu) \cong T$. Notice that $HI(C_\mu) \cong T$ if and only if $I(\pi_2^\mu)$ is an isomorphism in diagram (4), since $HI(T) \cong T$.

The following Lemma 4.1, which states a trivial result in sets, will be needed in the proofs of the "if parts" of Theorems 4.1 and 5.1.

Lemma 4.1. *Let gf be the composite of a pair $f : A \to B$, $g : B \to C$ of surjections in the category of sets. Consider the pullback $pr_1 : f^{-1}g^{-1}(\{c\}) \to A$ of the function $\hat{c} : \{*\} \to C$, $\hat{c}(*) = c$, along the function $gf : A \to C$, for any element $c \in C$ (see diagram (5) below). Then, the function g is an injection if and only if, for every element $c \in C$, $fw = \hat{b}!$ for some function $\hat{b} : \{*\} \to B$ (i.e., fw factorises through a one point set), where $!$ denotes the unique function into $\{*\}$.*

(5)
$$\begin{tikzcd}
f^{-1}g^{-1}(\{c\}) \arrow[rr, "pr_2"] \arrow[dd, "pr_1"'] \arrow[dr] & & \{*\} \arrow[dd, "\hat{c}"] \\
& \{*\} \arrow[d, "\hat{b}"] & \\
A \arrow[r, "f"'] & B \arrow[r, "g"'] & C
\end{tikzcd}$$

Theorem 4.1. *Under the assumptions of section 2, the full reflection $I \dashv H$ is semi-left-exact if and only if $HI(C_\mu) \cong T$, for every connected component C_μ, where T is any terminal object.*

Proof. If $I \dashv H$ is semi-left-exact then, by Definition 3.2, $I(C \times_{HI(C)} M)$ must be isomorphic to $I(M)$ in diagram (2), since $I(\eta_C)$ is an isomorphism.[4] In particular, $I(C \times_{HI(C)} M) \cong I(T)$ if $M \cong T$.

Suppose now that every connected component is connected, that is, $I(C_\mu) \cong T$ for every $\mu : T \to HI(C)$, $C \in \mathbb{C}$, and consider the diagram:

(6)
$$\begin{tikzcd}
C_{g\mu} \arrow[rrr, "pr_2"] \arrow[dd, "pr_1"'] \arrow[dr, "\eta_{C_{g\mu}}"] & & & T \arrow[dd, "\mu"] \\
& HI(C_{g\mu}) \arrow[dr, "HI(pr_1)"] & & \\
C \times_{HI(C)} M \arrow[rr, "\eta_{C \times_{HI(C)} M}"'] \arrow[d, "\pi_1"'] & & HI(C \times_{HI(C)} M) \arrow[r, "HI(\pi_2)"'] \arrow[d, "HI(\pi_1)"] & M \arrow[d, "g"] \\
C \arrow[rr, "\eta_C"'] & & HI(C) \arrow[r, "1_{HI(C)}"'] & HI(C)
\end{tikzcd}$$
.

[4] $\varepsilon_{I(C)} I(\eta_C) = 1_{I(C)}$, where $\varepsilon : IH \to 1_{\mathbb{M}}$ is the counit of the full reflection and therefore an isomorphism.

The bottom rectangle in diagram (6) is a pullback square of the form (2), since $HI(\pi_2)\eta_{C\times_{HI(C)}M} = \eta_M\pi_2$ and η_M is an identity, because $M \in \mathbb{M}$ (cf. remark 2.1). According to (a), (b) and (c) in section 2, the reflection $I \dashv H$ is semi-left-exact if and only if $UHI(\pi_2)$ is an injection in **Set**, in every diagram (6). The upper rectangle in diagram (6) (associated to the equation $\mu pr_2 = HI(\pi_2)\eta_{C\times_{HI(C)}M}pr_1$) is a pullback square, therefore the outer rectangle in diagram (6) is in fact a pullback square of the form (4), and $C_{g\mu}$ is the connected component associated to $g\mu : T \to HI(C)$. Then, as (d) in section 2 holds, by Lemma 4.1, $UHI(\pi_2)$ is an injection since every connected component is connected, in particular $HI(C_{g\mu}) \cong T$, for any morphisms $g : M \to HI(C)$, with $M \in \mathbb{M}$, and $\mu : T \to M$, with T terminal.

□

5. Stable Units Property and Product of Connected Components

Theorem 5.1. *Under the assumptions of section 2, the full reflection $I \dashv H$ has stable units if and only if $HI(C_\mu \times D_\nu) \cong T$, for every pair of connected components C_μ, D_ν, where T is any terminal object.*

Proof. If $I \dashv H$ has stable units then the functor I preserves finite products, since a product diagram is a pullback square in which the right corner in the bottom is a terminal object $T \in \mathbb{M}$ (cf. remark 3.3). Therefore, $HI(C_\mu \times D_\nu) \cong T$ since $HI(C_\mu) \cong T \cong HI(D_\nu)$, by Theorem 4.1, for every pair of connected components C_μ, D_ν.

Suppose now that every product of two connected components is connected, i.e., $HI(C_\mu \times D_\nu) \cong T$ for every pair of morphisms $\mu : T \to HI(C)$ and $\nu : T \to HI(D)$, $C, D \in \mathbb{C}$, and consider the diagram:

$$
(7)\qquad
\begin{tikzcd}
C_{HI(g)\nu} \times D_\nu \arrow[rrr, "p_2"] \arrow[ddd, "p_1"] \arrow[dr, "w"] & & & D_\nu \arrow[d, "\pi_1^\nu"] \\
& C \times_{HI(C)} D \arrow[rr, "\pi_2"] \arrow[dd, "\pi_1"] \arrow[dr, "\eta_{C\times_{HI(C)}D}"] & & D \arrow[d, "\eta_D"] \\
& & HI(C \times_{HI(C)} D) \arrow[r, "HI(\pi_2)"] \arrow[d, "HI(\pi_1)"] & HI(D) \arrow[d, "HI(g)"] \\
C_{HI(g)\nu} \arrow[r, "\pi_1^{HI(g)\nu}"'] & C \arrow[r, "\eta_C"'] & HI(C) \arrow[r, "1_{HI(C)}"'] & HI(C)
\end{tikzcd}
.
$$

The inside rectangle in diagram (7) is a pullback square of the form (3), since $HI(g)\eta_D = \eta_{HI(C)}g$ and $\eta_{HI(C)}$ is an identity, because $HI(C) \in \mathbb{M}$ (cf. remark 2.1).

According to (a), (b) and (c) in section 2, the reflection $I \dashv H$ has stable units if and only if $UHI(\pi_2)$ is an injection in **Set**, for every diagram of the form (3). In fact, $UHI(\pi_2)$ is obviously a surjection, since $UHI(\pi_2)U(\eta_{C\times_{HI(C)}D}) = U(\eta_D)U(\pi_2)$ and

$U(\eta_{C\times_{HI(C)}D})$, $U(\eta_D)$ and $U(\pi_2)$ are all surjections by the assumptions in section 2. The morphisms p_1 and p_2 in diagram (7) are the product projections of the product of the connected components $C_{HI(g)\nu}$ and D_ν. The morphism w is the unique morphism which makes diagram (7) commute; it is well defined since

$$\begin{gathered} HI(g)\eta_D\pi_1^\nu p_2 = HI(g)\nu\pi_2^\nu p_2 = \\ = HI(g)\nu\pi_2^{HI(g)\nu}p_1 \\ \text{(because both } \pi_2^\nu p_2 \text{ and } \pi_2^{HI(g)\nu}p_1 \text{ have the same domain and codomain,} \\ \text{the latter being the terminal object } T) \\ = \eta_C\pi_1^{HI(g)\nu}p_1. \end{gathered}$$

Then, as (d) in section 2 holds, by Lemma 4.1, $UHI(\pi_2)$ is an injection if the outer rectangle in the following diagram is a pullback square, for every morphism $\nu : T \to HI(D)$ from the terminal object into $HI(D)$ (cf. diagram (5)):

(8)
$$\begin{tikzcd}
C_{HI(g)\nu}\times D_\nu \arrow[r, "p_2"] \arrow[dr, "\eta_{C_{HI(g)\nu}\times D_\nu}"'] \arrow[dd, "w"'] & D_\nu \arrow[r, "\pi_2^\nu"] & T \arrow[dd, "\nu"] \\
 & HI(C_{HI(g)\nu}\times D_\nu) \arrow[d, "HI(w)"] & \\
C\times_{HI(C)} D \arrow[r, "\eta_{C\times_{HI(C)}D}"'] & HI(C\times_{HI(C)} D) \arrow[r, "HI(\pi_2)"'] & HI(D)
\end{tikzcd}.$$

In order to show that the outer rectangle in diagram (8) is a pullback square, consider a morphism $l : A \to C\times_{HI(C)} D$ such that $HI(\pi_2)\eta_{C\times_{HI(C)}D}l = \nu!$. Let $\bar{l} = \langle l_1, l_2\rangle : A \to C_{HI(g)\nu}\times D_\nu$ be the morphism into the product of the two connected components, in which $l_1 : A \to C_{HI(g)\nu}$ and $l_2 : A \to D_\nu$ are the morphisms determined in the pullback squares of the connected components by $\pi_1^{C_{HI(g)\nu}}l_1 = \pi_1 l$ and $\pi_1^{D_\nu}l_2 = \pi_2 l$, respectively. It is then a routine calculation to verify that w is a monomorphism and $w\bar{l} = l$.

□

Remark 5.1. It is an immediate consequence of Theorems 5.1 and 4.1 that, provided the preservation of finite products by the left adjoint I is added to the assumptions of section 2, the reflection $I \dashv H$ has stable units if and only if it is semi-left-exact.

6. Left-Exactness and Pullbacks of Connected Components

The following Theorem 6.1 gives a sufficient condition for the reflection $I \dashv H$ to be a localization, that is, for the left adjoint I to be left exact (see section 3).

Theorem 6.1. *Under the assumptions of section 2, the full reflection $I \dashv H$ is a localization if $HI(A_\mu \times_C B_\nu) \cong T$, for every pullback $A_\mu \times_C B_\nu$ of any pair of connected components A_μ, B_ν, where T is any terminal object. That is, the left adjoint is left exact if every pullback of connected components is connected.*

Proof. Consider the diagram

$$
\begin{tikzcd}
A_\mu \times_C B_\nu \arrow[rrrrr, "p_2"] \arrow[ddddd, "p_1"'] \arrow[dr, "j"] & & & & & B_\nu \arrow[d, "\pi_1^\nu"] \\
& A \times_C B \arrow[rrrr, "\pi_2"] \arrow[dddd, "\pi_1"'] \arrow[dr, "\eta_{A \times_C B}"] & & & & B \arrow[dddd, "g"] \arrow[dl, "\eta_B"'] \\
& & HI(A \times_C B) \arrow[rr, "HI(\pi_2)"] \arrow[dd, "HI(\pi_1)"'] \arrow[dr, "w"] & & HI(B) \arrow[dd, "HI(g)"] & \\
& & & HI(A) \times_{HI(C)} HI(B) \arrow[ur] \arrow[dl] & & \\
& & HI(A) \arrow[rr, "HI(f)"'] & & HI(C) & \\
A_\mu \arrow[r, "\pi_1^\mu"'] & A \arrow[rrrr, "f"'] \arrow[ur, "\eta_A"] & & & & C \arrow[ul, "\eta_C"']
\end{tikzcd}
\quad , \tag{9}
$$

wherein $A_\mu \times_C B_\nu = A_\mu \times_{(f\pi_1^\mu, g\pi_1^\nu)} B_\nu$ and $HI(A) \times_{HI(C)} HI(B) = HI(A) \times_{(HI(f), HI(g))} HI(B)$ are pullbacks, and j and w are the unique morphisms making the diagram commute.

One has to prove that $U(w)$ is always a bijection. It follows from $I(A_\mu \times_C B_\nu) \cong T$ that $U(A_\mu \times_C B_\nu) \neq \emptyset$, for all connected components A_μ, B_ν, which implies that $U(w)$ is a surjection, under the assumptions of section 2. Note that $U(\eta_{A\times_C B})^{-1} U(w)^{-1}(U(A_\mu), U(B_\nu)) = U(A_\mu \times_C B_\nu)$ in **Set**, which implies that $U(w)$ is an injection, since $UHI(j)U(\eta_{A_\mu \times_C B_\nu}) = U(\eta_{A\times_C B})U(j)$ and $UHI(A_\mu \times_C B_\nu) = \{*\}$. □

7. Admissibility of a Simple Reflection

Theorem 7.1. *Let the following condition and all assumptions of section 2 hold: (e) every map $I_{T,C} : \mathbb{C}(T, C) \to \mathbb{M}(T, I(C))$, the restriction of the reflector I to the hom-set $\mathbb{C}(T, C)$, is a surjection, for every object $C \in \mathbb{C}$, with $T = HI(T)$ a terminal object in $\mathbb{C}$. Then, the reflection $I \dashv H$ is semi-left-exact if and only if it is simple.*

Proof. Suppose that $I \dashv H$ is a simple reflection, that is, $I(w)$ is an isomorphism in every diagram of the form (1) in Definition 3.1, and consider the pullback square (4) in Definition 4.1. Let $w : T \to C_\mu$ be the unique morphism such that $\pi_1^\mu w = \nu$ and $\pi_2^\mu w = 1_T$, where ν is such that $HI(\nu) = \mu$ (ν exists by (e) in the statement).

Note that the composite $I(\pi_2^\mu)I(w)$ is the isomorphism 1_T. Therefore, $I(\pi_2^\mu)$ is an isomorphism, since $I(w)$ is an isomorphism by assumption.

□

8. Examples

1. Consider the full reflection of compact Hausdorff spaces into Stone spaces $H \vdash I : \mathbf{CompHaus} \to \mathbf{Stone}$, where each unit map $\eta_X : X \to HI(X)$ is the canonical projection of X into the set of its components, this set being given the quotient topology with respect to η_X. Hence, condition (c) in section 2 holds for the functor U which forgets the topology. Conditions (a) and (b) of section 2 hold as well since $U : \mathbf{CompHaus} \to \mathbf{Set}$ is monadic, and condition (d) holds trivially. This reflection is known to have stable units, therefore finite products of connected components are connected.

Let $\hat{0} : T \to [0,1]$ and $\hat{1} : T \to [0,1]$ be the two obvious inclusions of the one point topological space into the closed interval of real numbers $[0,1]$, with the usual topology. Then, the pullback $T \times_{(\hat{0},\hat{1})} T = \emptyset$ is the empty space, not connected in our sense, being clear that the reflector I is not left exact, since it does not preserve the pullback diagram of $\hat{0}$ and $\hat{1}$, and also that the sufficient condition of Theorem 6.1 does not hold.

2. With the exception of (d), every assumption of section 2 hold for any reflection from a variety of universal algebras into one of its subvarieties, provided with the forgetful functor into **Set**. Notice that, for these reflections, condition (d) of section 2 is equivalent to idempotency of the algebras in the subvariety, meaning that every element of an algebra in the subvariety is a subalgebra.

In particular, it is easy to check that condition (d) in section 2 holds for the reflection $H \vdash I : \mathbf{SGr} \to \mathbf{SLat}$ of semigroups into semilattices, which is known to have stable units (see [3]). Therefore, all finite products of connected components are connected.

The additive semigroup $\mathbb{N}$ of non-negative integers has two connected components, $\{0\}$ and $\{1,2,3,...\}$, with respect to the reflection $\mathbf{SGr} \to \mathbf{SLat}$. The pullback of the inclusions $\{0\} \to \mathbb{Z}$ and $\{1,2,3,...\} \to \mathbb{Z}$ into the integers is the empty semigroup $\emptyset$, which is not connected ($I(\emptyset) = \emptyset$ is not terminal). Hence, this reflection is not a localization, and also the sufficient condition of Theorem 6.1 does not hold.

The reflection $H \vdash I : \mathbf{SGr} \to \mathbf{Band}$ of semigroups into bands[5] is not a semi-left-exact reflection (cf. [3]). Notwithstanding, all assumptions in section 2 hold for this reflection; therefore not every connected component is connected, by Theorem 4.1 (see Example 7 in [3]).

Note that Theorem 7.1 holds for the reflection $H \vdash I : \mathbf{Band} \to \mathbf{SLat}$ from bands into semilattices (a subreflection of $\mathbf{SGr} \to \mathbf{SLat}$).[6]

[5]A semigroup is called a band if every one of its elements is idempotent.

[6]Remark that in algebraic instances 2., condition (d) in the ground structure is crucial, while condition (b) is the crucial one in the former topological instances 1.

3. Finally, we would like to remark that the joining of new *geometrical* examples, to the *algebraic* and *topological* well-known examples above, has been made possible by a generalization of the assumptions in the ground structure, done in [6], where a new class of instances is presented.

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